On the Acceleration of a Superconducting Macroparticle in a Magnetic Travelling Wave Accelerator

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Two models for describing the operation of a magnetic travelling wave accelerator, the transmission line model and the lumped parameter model, are analyzed. For both cases the current distribution and the magnetic field are calculated. It is shown that the transmission line model can be obtained as the limiting case of the lumped parameter model. It is then assumed that a superconducting solenoid with large persistent currents is used as a projectile with a constant magnetic dipole moment. The motion of this projectile is studied in the case of ignorable flux interaction. The conditions for phase and rotational stability are obtained. It is shown that the projectile has a transverse translational instability. A derivation of the flux interaction current wave is included in the Appendix.

I. Introduction

Several years ago it was suggested that controlled nuclear fusion might be achieved by using hypervelocity (10^8 cm/sec) projectiles [1, 2]. Bullets with velocities in the range of 10^6 to 10^8 cm/sec might also be used as research tools in the production of artificial meteors and in the study of dense plasmas.

It has been shown furthermore by D. J. ROSE (unpublished communication) that the replenishment of thermonuclear fuel into conventional fusion devices can be most expediently accomplished by the injection of solid T – D pellets into the magnetically confined hot plasma region. It was shown that the required velocities to avoid substantial ablation of the pellet during injection are of the order of 10^6 cm/sec and are thus larger by one order of magnitude than in conventional rifles.

Various devices for accelerating macroscopic particles to hypervelocities have been tried or proposed. Among these have been the bench-mounted rifle, light-gas gun, shaped charge, rocket, electrostatic accelerator, induction accelerator, and magnetic accelerator. It can be shown that none of these methods give final velocities in the desired range to projectiles of large mass (~ 1 gm). It has been suggested that very high acceleration could be obtained by injecting a superconducting projectile into a magnetic travelling wave device [3–5]. The high currents possible in type II superconductors may be sustained under the strong magnetic fields required to obtain an acceleration of 10^{10} cm/sec^2 (l. c. [6]). A schematic diagram of the magnetic travelling wave accelerator is shown in Fig. 1.1.

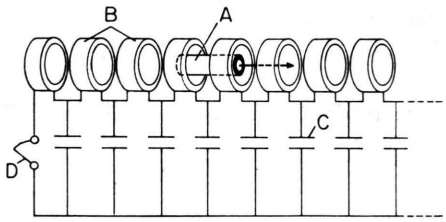

Fig. 1.1. Magnetic travelling wave accelerator. A: accelerated solenoid. B: field coils. C: capacitors. D: switch to be closed to start magnetic travelling wave.

Two simple models, the continuous transmission line model (TLM) and the related lumped parameter model (LPM), are studied herein. In both cases the accelerator is assumed to have negligible mutual inductance. A parallel wire transmission line has no "coil-coil" interaction because of symmetry. For a practical accelerator this interaction would

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[1] F. WINTERBERG, Case Institute of Technology Plasma Research Program Technical Report No. A 21, June 1963, and Z. Naturforsch. **19 a**, 231 [1964].

[2] E. R. HARRISON, Phys. Rev. Letters **11**, 535 [1963].

[3] C. MAISONNIER, Nuovo Cim. **42 B**, 332 [1966].

[4] F. WINTERBERG, Nucl. Fusion **6**, 152 [1966].

[5] F. WINTERBERG, J. Nucl. Energy, Part C, **8**, 541 [1966].

[6] Tensile strengths place an upper limit of about 10^{10} cm/sec^2 on the attainable accelerations. If 10^8 cm/sec is the final projectile velocity, then this acceleration corresponds to an accelerator length of 5 km.

contribute to the damping of the travelling wave. The projectile is assumed to be a superconducting solenoid with large persistent currents making it a permanent magnetic dipole. It is also assumed that the applied magnetic fields do not upset the superconducting state of the projectile. The flux interaction between the projectile's magnetic field and the accelerating coils is considered ignorable; the current due to flux interaction in the transmission line model is derived in the Appendix.

In case such a macro-particle accelerator is applied for the production of dense thermonuclear plasmas, very large accelerator dimensions will result. If the more modest goal of reaching meteoric velocities is envisaged, the accelerator dimensions are still large but well within the limits of reasonable expenditure. For the use of such an accelerator system in the thermonuclear fuel injection problem, the accelerator length is well within the realm of technical feasibility and in the order of several meters. In this latter case though, the superconducting projectile would act as a catapult to inject a solid piece of T − D into the thermonuclear device, by placing the T − D pellet in front of the projectile, to be released by the catapult. In this mode of operation the superconducting projectile serving as the catapult would be slowed down after reaching its maximum speed. The slowing down of the superconducting projectile can there be accomplished by the same principle of projectile field-coil magnetic interaction.

II. The Transmission Line Model (TLM)

A diagram of the basic circuit in the TLM is shown in Fig. 2.1. The inductance per unit length, $L_1(x)$, and the capacitance per unit length, $C_1(x)$, are continuous functions of the position along the accelerator. The line is initially charged up to a potential differenc of V_0 between the upper and lower wires. At time $t=0$ the switch is closed and a current wave moves to the right. This current wave generates the magnetic travelling wave which accelerates the projectile.

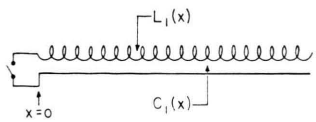

Fig. 2.1.

For the case in which the accelerator currents are not significantly perturbed by the projectile, the ordinary transmission line equations hold[7]:

$$L_1(\partial I/\partial t) + R_1 I = -\partial V/\partial x\,, \qquad (2.1)$$

$$C_1(\partial V/\partial t) + G_1 V = -\partial I/\partial x\,. \qquad (2.2)$$

In these equations R_1 and G_1 are, respectively, the resistance and leakage conductance per unit length. We assume that the damping effects caused by R_1 and G_1 are small. If superconducting coils of the high field type are used for the accelerator coils, this approximation becomes very good. Putting $R_1 = G_1 = 0$ and taking the case in which L_1 and C_1 are constants, we obtain from Eqs. (2.1) and (2.2)

$$\frac{\partial^2 I}{\partial x^2} - L_1 C_1 \frac{\partial^2 I}{\partial t^2} = 0\,. \qquad (2.3)$$

The above one dimensional wave equation has solutions of the form

$$I = I(x \pm v\,t)\,, \qquad (2.4)$$

where the wave propagation velocity v is given by

$$v = 1/\sqrt{L_1 C_1}\,. \qquad (2.5)$$

This expression suggests that it should be possible to obtain an accelerating wave by varying $L_1(x)$ and $C_1(x)$ as functions of x. A point on the wave would move as

$$x = \tfrac{1}{2}\,a\,t^2 + v_0\,t\,, \qquad (2.6)$$

$$t = \frac{1}{a}\left(-v_0 + \sqrt{v_0^2 + 2\,a\,x}\right), \qquad (2.7)$$

where a is the constant acceleration of the wave and v_0 is its initial velocity. From Eq. (2.5) and the time derivative of Eq. (2.6) we thus would obtain

$$L_1(x)\,C_1(x) = 1/(v_0^2 + 2\,a\,x)\,. \qquad (2.8)$$

The general solution, Eq. (2.4), depends on L_1 and C_1 being constants. The particular form of this solution depends on the initial conditions. As suggested earlier, we set

$$V(x,0) = V_0 \quad \text{for} \quad x > 0\,, \qquad (2.9)$$

$$V(0,t) = 0 \quad \text{for} \quad t \geqq 0\,. \qquad (2.10)$$

Since L_1 and C_1 are constants, Eq. (2.3) is easily solved by the methods of operational calculus. Denoting the Laplace transform of $I(x,t)$ by $i(x,p)$, we have for the transform of Eq. (2.3)

$$i_{xx} - p^2\,\mathrm{L}_1\,C_1\,i = 0\,, \qquad (2.11)$$

[7] W. T. Scott, The Physics of Electricity and Magnetism, John Wiley & Sons, New York 1966, p. 532.

where the subscripts on i denote differentiation with respect to x. This equation has the solution

$$i(x,p) = b_1 e^{px\sqrt{L_1C_1}} + b_2 e^{-px\sqrt{L_1C_1}}, \quad (2.12)$$

where b_1 and b_2 are arbitrary constants. Applying the initial conditions to this gives

$$i(x,p) = -C_1 V_0 p^{-1}(L_1 C_1)^{-1/2} e^{-px\sqrt{L_1C_1}}, \quad (2.13)$$

which has the Laplace inversion

$$I(x,t) = -V_0\left(\frac{C_1}{L_1}\right)^{1/2} H(t - x\sqrt{L_1 C_1}), \quad (2.14)$$

where H is the Heaviside step function. The current wave form is shown in Fig. 2.2. Differentiating the wave front position $x = t/\sqrt{L_1 C_1}$ with respect to time, we obtain $v = 1/\sqrt{L_1 C_1}$, which is consistent with Eq. (2.5).

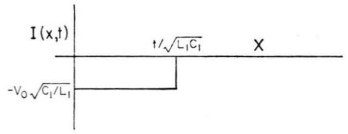

Fig. 2.2.

In the more general case in which $L_1(x)$ and $C_1(x)$ are slowly varying functions of x, the WKB method can be applied. The generalization of Eq. (2.3) obtained from Eqs. (2.1) and (2.2) can be written as

$$I_{xx} - I_x(\ln C_1)_x - L_1 C_1 I_{tt} = 0. \quad (2.15)$$

Taking the time Laplace transform of this results in

$$i_{xx}(x,p) - i_x(x,p)(\ln C_1)_x - p^2 L_1 C_1 i(x,p) = 0. \quad (2.16)$$

For slowly varying $L_1(x)$ and $C_1(x)$, the two independent WKB solutions are of the form

$$i(x,p) = b_1 C_1^{1/2}(p^2 L_1 C_1)^{-1/4} \exp\{\int_0^x (p^2 L_1 C_1)^{1/2} dx\} + b_2 C_1^{1/2}(p^2 L_1 C_1)^{-1/4} \exp\{-\int_0^x (p^2 L_1 C_1)^{1/2} dx\}. \quad (2.17)$$

From the initial conditions we find

$$i(x,p) = -V_0 p^{-1}\left(\frac{C_1(0)\; C_1(x)}{L_1(0)\; L_1(x)}\right)^{1/4} \exp\{-p\int_0^x (L_1 C_1)^{1/2} dx\}. \quad (2.18)$$

The Laplace inversion is formally the same as before; we obtain for the current distribution

$$I(x,t) = -V_0\left(\frac{C_1(0)\; C_1(x)}{L_1(0)\; L_1(x)}\right)^{1/4} H\left(t - \int_0^x (L_1 C_1)^{1/2} dx\right), \quad (2.19)$$

which reduces properly to Eq. (2.14) when L_1 and C_1 are constants. Here again, by differentiating the wave front position $\int_0^x (L_1 C_1)^{1/2} dx = t$ with respect to time we obtain $dx/dt = v = 1/\sqrt{L_1 C_1}$, as in Eq. (2.5) which was derived for constant $L_1 C_1$. This then confirms the correctness of Eq. (2.8) for constant acceleration.

Before considering the magnetic travelling wave and the associated force field, we calculate the current distribution for the LPM.

III. The Lumped Parameter Model (LPM)

In the preceding model (TLM) the quantities L_1 and C_1 were measured per unit length. In the model we now consider (LPM) L and C are measured per coil, or per "lump". Figure 3.1 indicates the design of the LPM. The accelerator tube is contained inside the several coils placed end to end. This design is somewhat more realistic than the TLM, since it is impractical to construct an accelerator with continuous parameters.

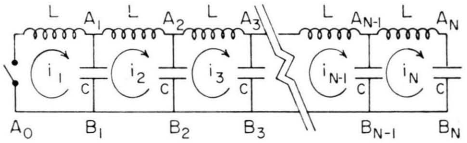

Fig. 3.1.

The initial conditions are the same as before. All the capacitors are initially charged to a voltage V_0, and at the time $t = 0$ the switch is closed. If $E_{A_n B_n}$ denotes the voltage at point B_n relative to point A_n, and if i_n represents the current in the n-th coil, then the initial conditions can be expressed as

$$\{E_{A_n B_n} = -V_0\}: \; n = 1, 2, \ldots, N, \quad (3.1)$$

$$\{i_n = 0\}: \; n = 1, 2, \ldots, N. \quad (3.2)$$

Using the differential equation for an inductor, one finds

$$\left\{E_{A_{n-1} A_n} = -L\frac{di_n}{dt}\right\}: \quad n = 1, 2, \ldots, N. \quad (3.3)$$

From Eqs. (3.1) and (3.3) one may then write

$$\left\{\frac{di_n}{dt}\bigg|_0 = \delta_{n1}\left(\frac{-V_0}{L}\right)\right\}: \quad n = 1, 2, \ldots, N. \quad (3.4)$$

From the differential equation for a capacitor one has the set of equations

$$\left\{i_n - i_{n+1} = -C\frac{d}{dt}(E_{A_n B_n})\right\}: \quad n = 1, 2, \ldots, N. \quad (3.5)$$

Since the voltage drops along a line are additive, we have

$$\{E_{\mathrm{A}_n \mathrm{B}_n} = E_{\mathrm{A}_n \mathrm{A}_0} = -\sum_{k=1}^{n} E_{\mathrm{A}_{k-1} \mathrm{A}_k}\} : n = 1, 2, \dots, N . \tag{3.6}$$

Combining these equations results in the set

$$\left\{ L C \sum_{k=1}^{n} \frac{\mathrm{d}^2 i_k}{\mathrm{d}t^2} + i_n - i_{n+1} = 0 \right\} : \quad n = 1, 2, \dots, N . \tag{3.7}$$

A simpler form is obtained by putting $t = s\sqrt{L C}$; the set of equations, including the initial conditions, can then be written as

$$\left\{ \sum_{k=1}^{n} \frac{\mathrm{d}^2 i_k}{\mathrm{d}s^2} + i_n - i_{n+1} = 0 \right\} : \quad n = 1, 2, \dots, N , \tag{3.8}$$

$$\left\{ \left. \frac{\mathrm{d} i_n}{\mathrm{d}s} \right|_0 = \delta_{n1} U_0 \right\} : \quad n = 1, 2, \dots, N , \tag{3.9}$$

where $U_0 = -V_0 \sqrt{C/L}$. Applying the Laplace transform to the system results in

$$\{p^2 \sum_{k=1}^{n} a_k(p) + a_n(p) - a_{n+1}(p) = U_0\} : n = 1, 2, \dots, N , \tag{3.10}$$

where $a_n(p)$ is the transform of $i_n(s)$. Using Kramer's rule to obtain the solution, we write

$$a_n(p) = D_{\mathrm{H}}^n / D_{\mathrm{H}} , \tag{3.11}$$

where D_{H} is the determinant of the coefficient matrix of the homogeneous part of Eq. (3.10) and D_{H}^n is the "adjusted" determinant of the coefficient matrix. The simplest forms can be obtained by using some of the theorems of determinant algebra.

Interchange of the rows or columns of a determinant merely changes the sign of the determinant, and subtraction of an arbitrary row from any other row leaves the determinant unchanged. These N by N determinants can be written as

$$D_{\mathrm{H}} = \begin{vmatrix} p^2+1 & -1 & 0 & \cdots & & & \\ -1 & p^2+2 & -1 & \cdots & & \mathrm{O} & \\ 0 & -1 & p^2+2 & \cdots & & & \\ \vdots & \vdots & \vdots & & & \vdots & \vdots \\ & & & & & -1 & 0 \\ & \mathrm{O} & & \cdots & -1 & p^2+2 & -1 \\ & & & \cdots & 0 & -1 & p^2+2 \end{vmatrix} \; (N \times N) \text{ det} , \tag{3.12}$$

and

$$D_{\mathrm{H}}^n = \begin{vmatrix} U_0 & p^2+1 & -1 & 0 & \cdots & & \text{col. } n & \text{col. } n+1 & & & & & \\ 0 & -1 & p^2+2 & -1 & \cdots & & & & & & \mathrm{O} & & \\ 0 & 0 & -1 & p^2+2 & \cdots & & \vdots & \vdots & \vdots & & & & \\ \vdots & \vdots & \vdots & \vdots & & & -1 & 0 & 0 & & & & \\ & & & & \cdots & -1 & p^2+2 & 0 & 0 & \cdots & & & \\ & & & & \cdots & 0 & -1 & -1 & 0 & \cdots & & & \\ & & & & \cdots & 0 & 0 & p^2+2 & -1 & \cdots & & & \\ & & & & \cdots & 0 & 0 & -1 & p^2+2 & \cdots & & \vdots & \vdots \\ & & & & & \vdots & \vdots & \vdots & \vdots & & & -1 & 0 \\ & & \mathrm{O} & & & & & & \cdots & & -1 & p^2+2 & -1 \\ & & & & & & & & \cdots & & 0 & -1 & p^2+2 \end{vmatrix} \; \begin{matrix} (N \times N) \text{ det} \\ \\ (-1)^{n-1} . \end{matrix} \tag{3.13}$$

Using the definition

$$\Delta_n = \begin{vmatrix} p^2+2 & -1 & 0 & \cdots & & & \\ -1 & p^2+2 & -1 & \cdots & & \mathrm{O} & \\ 0 & -1 & p^2+2 & \cdots & & & \\ \vdots & \vdots & \vdots & & & \vdots & \vdots \\ & & & & & -1 & 0 \\ & \mathrm{O} & & \cdots & -1 & p^2+2 & -1 \\ & & & \cdots & 0 & -1 & p^2+2 \end{vmatrix} \; (n \times n) \text{ det} , \tag{3.14}$$

the above determinants can be written in the compact forms

$$D_{\mathrm{H}} = (p^2+1)\, \Delta_{N-1} - \Delta_{N-2} , \tag{3.15}$$

and

$$D_{\mathrm{H}}^n = U_0\, \Delta_{N-n} . \tag{3.16}$$

For Δ_N we have the recursion relationship

$$\Delta_N = (p^2+2)\, \Delta_{N-1} - \Delta_{N-2} , \tag{3.17}$$

which is of the same form [8] as the hyperbolic identity

$$\sinh[N u] = 2 \cosh[u] \sinh[(N-1) u] - \sinh[(N-2) u] . \tag{3.18}$$

Defining u by the relationship

$$2 \cosh u = p^2 + 2 = \Delta_1 , \tag{3.19}$$

it follows by induction that

$$\Delta_N = \frac{\sinh[(N+1) u]}{\sinh u} . \tag{3.20}$$

Combining Eqs. (3.15) and (3.17), one obtains

$$D_{\mathrm{H}} = \Delta_N - \Delta_{N-1} , \tag{3.21}$$

or, from Eq. (3.20),

$$D_{\mathrm{H}} = \frac{\sinh[(N-1) u] - \sinh[N u]}{\sinh u} . \tag{3.22}$$

The zeros of D_{H} are of interest since, through Eq. (3.11), they are the poles of $a_n(p)$. Values of u

[8] W. T. SCOTT, Phys. Rev. **76**, 212 [1949].

giving all the zeros of D_H are

$$u_l = i\beta_l = i\left(\frac{2l+1}{2N+1}\right)\pi: \quad 0 \leqq l \leqq N-1. \qquad (3.23)$$

The corresponding poles in the p-plane are obtained from Eqs. (3.19) and (3.23). These poles, shown in Fig. 3.2, are given by

$$p_l^{\pm} = \pm i\omega_l = \pm 2i \sin\left[\frac{\pi}{2}\left(\frac{2l+1}{2N+1}\right)\right]. \qquad (3.24)$$

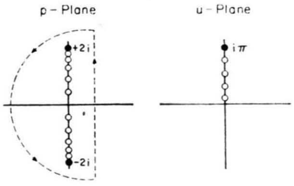

Fig. 3.2.

From the definition of the inverse Laplace transform the current solutions are

$$i_n = \frac{1}{2\pi i}\int_{-i\infty+\varepsilon}^{i\infty+\varepsilon} e^{ps}\, a_n(p)\, dp. \qquad (3.25)$$

This contour integral can be evaluated through use of the residue theorem, with the result being

$$i_n = \sum_l R_l, \qquad (3.26)$$

where R_l represent the residues of the pair of poles at $p_l^{\pm}$. Since none of the roots $p_l^{\pm}$ are degenerate, the poles of $a_n(p)$ are of first order. From the first order pole formula for residues it follows that

$$R_l = \lim_{p\to p_l^+}\left[\frac{(p-p_l^+)\, e^{ps}\, D_H{}^n}{D_H}\right] + \lim_{p\to p_l^-}\left[\frac{(p-p_l^-)\, e^{ps}\, D_H{}^n}{D_H}\right], \qquad (3.27)$$

where $D_H^n(i\omega)$ and $D_H(i\omega)$ are given functions of ω; hence

$$R_l = -2(\sin\omega_l s)\, D_H^n(i\omega_l) \lim_{\omega\to\omega_l}\left[\frac{(\omega-\omega_l)}{D_H(i\omega)}\right]. \qquad (3.28)$$

From Eqs. (3.22), (3.23), and (3.24), we obtain

$$\lim_{\omega\to\omega_l}\left[\frac{(\omega-\omega_l)}{D_H(i\omega)}\right] = \lim_{\beta\to\beta_l}\left[\frac{\sin\beta\,[\sin(\beta/2)-\sin(\beta_l/2)]}{\sin(\beta/2)\,\cos[(2N+1)\,\beta/2]}\right]. \qquad (3.29)$$

This limit can be taken by expanding $\sin(\beta/2)$ and $\cos[(2N+1)\,\beta/2]$ about $\beta=\beta_l$ to first order in a Taylor series. In this way R_l is obtained exactly.

Carrying out the summation indicated in Eq. (3.26) yields

$$i_n(s) = \sum_{l=0}^{N-1} \frac{U_0}{N+\frac{1}{2}} \sin\left[2s\sin\left(\frac{\pi}{2}\,\frac{2l+1}{2N+1}\right)\right] \times\cos\left[(n-\tfrac{1}{2})\,\frac{2l+1}{2N+1}\,\pi\right]\cot\left[\frac{\pi}{2}\,\frac{2l+1}{2N+1}\right]. \qquad (3.30)$$

The current distributions for an accelerator with a hundred "lumps" are shown in Fig. 3.3 for various values of the non-dimensional time s. Note that the behaviour of the current distributions is approximately that of a Heaviside wave.

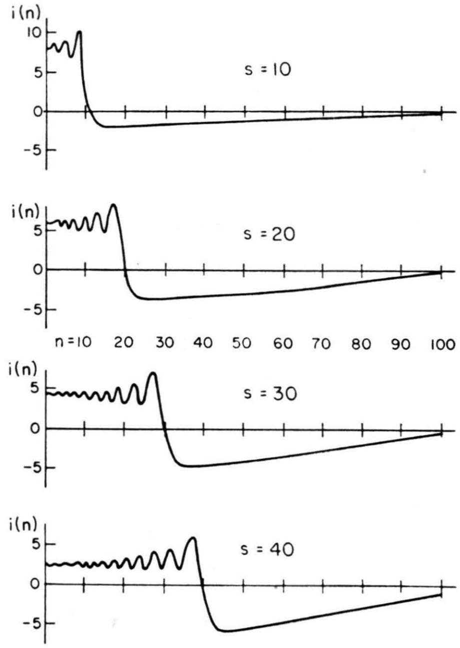

Fig. 3.3.

IV. Continuous Infinite Length Limit of the LPM

The two systems of units for inductance and capacitance are related by the equations

$$L_1 = kL, \qquad (4.1)$$

$$C_1 = kC. \qquad (4.2)$$

which implies that the nondimensional time becomes

$$s_1 = s/k, \qquad (4.3)$$

where k is the number of lumps per unit length. Using

$$w_l = \frac{2l+1}{2N+1}\,\frac{\pi}{2} \qquad (4.4)$$

and

$$\Delta w = \frac{\pi}{2}\,\frac{1}{N+\frac{1}{2}}, \qquad (4.5)$$

we obtain from Eq. (3.30), in the limit as $N\to\infty$, the integral

$$i(n, s_1) = \int_0^{\pi/2} dw\, \frac{2U_0}{\pi} \sin[2ks_1\sin(w)] \cdot\cos[(2n+1)\,w]\cot(w). \qquad (4.6)$$

Taking the limit as $k \to \infty$ results in the infinite length, continuous limit of the result Eq. (3.30), i. e.

$$i(x, s_1) = \frac{U_0}{2\pi} \int_{-\infty}^{\infty} \frac{dw}{w} \left[\sin(w - s_1)\, w - \sin(x - s_1)\, w\right] \tag{4.7}$$

which is just the Fourier integral expression for the Heaviside step function. Thus we have

$$i(x, t) = -V_0 (C_1/L_1)^{1/2} H(t - x L_1^{1/2} C_1^{1/2}), \tag{4.8}$$

which is identical to Eq. (2.14).

V. Phase Stability of the Acceleration

By phase stability is meant that the projectile remains ahead of the maximum peak of the magnetic travelling wave. The continuous infinite limit for the LPM has been shown to be just the TLM. For the LPM we may, therefore, consider the effective force field on the projectile to be a small, time dependent perturbation of the force field obtained for the TLM. For the TLM it is easy to establish phase stability, since there is an equilibrium position for the projectile in the co-moving frame of the current wave front. This co-moving frame of the TLM will also be the frame in which we study the LPM. The current wave front for the LPM oscillates about the current wave front of the TLM. Also, zero force in the co-moving frame will correspond to ideal uniform acceleration in the laboratory frame.

In order to carry out the stability analysis, an accelerator design must be chosen which specifies the spacing of the lumps as well as gives the desired acceleration. Two basic designs for the LPM will be considered, the time periodic case and the space periodic case. In the time periodic case, L and C are constants, but the spacings k^{-1} increase along the length of the accelerator. Its current wave distribution, Eq. (3.30), has a constant rate of propagation in the variable n; the current wave is made to accelerate in space by varying the spacings. Since we desire uniform acceleration, it is required that

$$k(x) = (L\,C)^{-1/2} (v_0^2 + 2\,a\,x)^{-1/2}. \tag{5.1}$$

The velocity, however, is given by

$$v(x) = (v_0^2 + 2\,a\,x)^{1/2} = k^{-1}(x)\,(L\,C)^{-1/2}, \tag{5.2}$$

which is proportional to the spacing. Thus the projectile passes equal numbers of lumps in equal intervals of time, hence the term "time periodic".

In the space periodic case we require that all the spacings be equal. In order for acceleration to occur, the value of $L\,C$ must vary from lump to lump. The fact that Eq. (3.30) is not valid for $L\,C$ variable is of no consequence, since the following analysis does not depend on the exact form of the current distribution.

We have chosen to use the space periodic case rather than the time periodic case for the following reason. A lump spacing that is required to increase with velocity would cause increasingly large fluctuations in the field seen by the projectile. If the lump spacing were to follow the velocity through several factors of ten, the projectile would soon fall behind the wave front in one of the weak field regions.

To calculate the force field from an actual current distribution would be unnecessarily difficult and exacting for the present treatment. Instead, we assume an effective current distribution which gives approximately the correct force field in the neighborhood of the projectile. This effective current distribution is a uniform solenoid of current, the leading edge of which oscillates back and forth about the ideal current wave front of the TLM. Oscillation occurs at the frequency at which the wave front passes lumps along the accelerator. If the projectile is sufficiently far ahead of the actual wave front, then the effective wave front produces the same local force field near the projectile. In other words, when the projectile is far enough away, it cannot distinguish between the vertical wave front of the effective wave and the sloped wave front of the actual current wave. The three types of current waves are compared in Fig. 5.1.

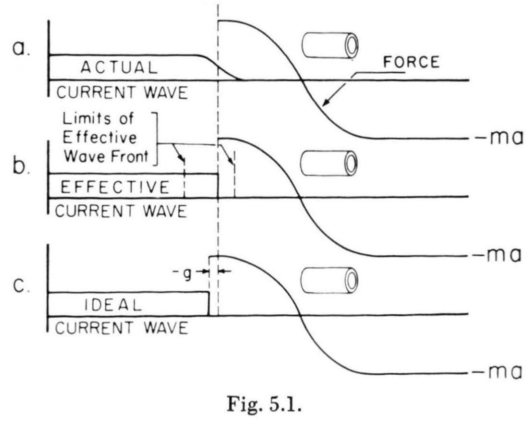

Fig. 5.1.

To describe the effective case quantitatively, we introduce the "bumpiness" variable g, the displacement of the effective current wave front from the ideal current wave front of the TLM. For the TLM and the effective case, the current wave is that of a uniform solenoid. Near the z-axis, the z-component of the magnetic field $\boldsymbol{B}$ is given by

$$B_z = \frac{2 I \pi}{c} \left[1 - z (R^2 + z^2)^{-1/2}\right], \qquad (5.3)$$

where R is the solenoid radius and z is the distance from the end of the solenoid.

Since the projectile is assumed to have a constant magnetic dipole moment, we calculate the force on a dipole aligned along the z-axis. To first order, one has

$$\boldsymbol{F} \cong \nabla (\boldsymbol{M} \cdot \boldsymbol{B}). \qquad (5.4)$$

In the co-moving frame the inertial force $-m\,a$ must be included, so

$$\boldsymbol{F} = \hat{k}\, M \frac{dB_z}{dz} - m\, a\, \hat{k}. \qquad (5.5)$$

Using Eq. (5.3), we obtain

$$m\,\ddot{h} = F_z = -m\,a - 2\,I\,M\pi\,R^2\,c^{-1}\,(R^2 + z^2)^{-3/2}, \qquad (5.6)$$

where $\ddot{h}$ is the acceleration in the co-moving frame. The distance from the leading edge of the solenoid to the projectile is

$$z = g + l_0 + h, \qquad (5.7)$$

where l_0 is the equilibrium distance between projectile and current wave for the TLM, h is the projectile's displacement from the ideal equilibrium point, and g is the bumpiness. We require that the effective wave oscillates about the ideal wave in such a way that the time average of g is zero,

$$\overline{g(t)} = 0. \qquad (5.8)$$

From Eqs. (5.6) and (5.7) we have

$$m\,\ddot{h} = -m\,a + A_1 [R^2 + (l_0 + \lambda)^2]^{-3/2}, \qquad (5.9)$$

where we have put

$$A_1 \equiv -2\,M\,I\,\pi\,R^2\,c^{-1}, \qquad \lambda \equiv h + g. \qquad (5.10)$$

A minimum criterion for stability is $A_1 > 0$. This corresponds to the existence of an equilibrium point in the ideal case for the TLM. If the projectile is to be well ahead of the region of the current wave front, we must have $l_0 \gg \lambda$. This allows us to expand Eq. (5.9) in a Taylor series in λ around $\lambda = 0$. To first order we obtain

$$m\,\ddot{h} = -m\,a + A_1 (R^2 + l_0^2)^{-3/2} - 3\,A_1\,l_0 (R^2 + l_0^2)^{-5/2}\,\lambda. \qquad (5.11)$$

The first two terms on the right drop out, however, since

$$-m\,a + A_1 (R^2 + l_0^2)^{-3/2} = 0. \qquad (5.12)$$

This can be seen by first noting that the conditions under which l_0 is defined demand that

$$g = \ddot{h} = h = 0, \qquad (5.13)$$

and then substituting into Eq. (5.9). From Eqs. (5.11) and (5.12) we obtain the general form of the equation of motion for the projectile in the co-moving frame,

$$m\,\ddot{h} + 3\,A_1\,l_0 (R^2 + l_0^2)^{-5/2} (h + g) = 0. \qquad (5.14)$$

Using Eq. (5.12) to define A_1 in terms of l_0, and setting

$$D \equiv 3\,m\,a\,l_0 (R^2 + l_0^2)^{-1}, \qquad (5.15)$$

we have from Eq. (5.14) that

$$m\,\ddot{h} + D\,h = -D\,g. \qquad (5.16)$$

The homogeneous form of this least equation is obtained by letting $g = 0$,

$$m\,\ddot{h} + D\,h = 0. \qquad (5.17)$$

Again, this is just the equation for the ideal case or the TLM. Its solution and characteristic frequency are given by

$$h = h_0 \cos[(D/m)^{1/2} (t - t_0)], \quad \omega_T = (D/m)^{1/2}. \qquad (5.18)$$

Eq. (5.16), which is the same as for a forced harmonic oscillator, is more difficult to solve. Its solution is readily obtainable only if $g(t)$ is a periodic function. In a space periodic accelerator, g would not be exactly time periodic. However, it might be sufficiently close to periodic that over a short interval of time its non-periodicity would produce only negligible small effects. To investigate this possibility we introduce the variable u, which gives the position of the ideal wave front as measured in units of length equal to the spacing k^{-1},

$$u = k (v_0\,t + a\,t^2/2). \qquad (5.19)$$

The bumpiness, g, is a periodic function of u, since the spacings are equal. Choosing to study the stability near the arbitrary time t_1, we put

$$\begin{aligned} t &= t_1 + \tau, \\ v_1 &= v_0 + a\,t_1, \\ x_1 &= v_0\,t_1 + a\,t_1^2/2. \end{aligned} \qquad (5.20)$$

Then we have

$$u = k (x_1 + v_1\,\tau + a\,\tau^2/2). \qquad (5.21)$$

For those cases in which the $a\,\tau^2/2$ term is negligibly small, we obtain g as a periodic function of τ.

Given values for a and v_1, there corresponds an upper limit τ_{max} for which this approximation is good. This is called the constant velocity approximation. The number of periods or lumps over which the constant velocity approximation is valid is given by

$$(\Delta u)_{max} = k\, v_1\, \tau_{max} = k\, v_1^2\, a^{-1}. \qquad (5.22)$$

We shall require only that $k \gg a/v_1^2$, so that $g(t)$ is periodic over the time interval $[-1/(k\,v_1), 1/(k\,v_1)]$. The $g(t)$ can then be represented by a Fourier series.

The solution to Eq. (5.16) can also be expanded in Fourier series. We put

$$g = \sum \alpha_n \sin(2\,\pi\, n\, k\, v_1\, \tau) + \sum \beta_n \cos(2\,\pi\, n\, k\, v_1\, \tau), \qquad (5.23)$$

$$h = \sum a_n \sin(2\,\pi\, n\, k\, v_1\, \tau) + \sum b_n \cos(2\,\pi\, n\, k\, v_1\, \tau). \qquad (5.24)$$

After obtaining h from Eq. (5.24), we substitute it into Eq. (5.16) and set the separate coefficients of the orthogonal functions equal to zero, with the result

$$a_n = \alpha_n\, D[m(2\,\pi\, n\, k\, v_1)^2 - D]^{-1}, \qquad (5.25)$$

$$b_n = \beta_n\, D[m(2\,\pi\, n\, k\, v_1)^2 - D]^{-1}. \qquad (5.26)$$

If one chooses values for the parameters such that

$$D \ll m(2\,\pi\, k\, v_1)^2, \qquad (5.27)$$

then, 1) the resonance conditions

$$D - m(2\,\pi\, n\, k\, v_1)^2 = 0 \qquad (5.28)$$

will be avoided and, 2) the amplitudes of a_n and b_n will be much smaller than those of α_n and β_n respectively. This insures that the Fourier coefficients are bounded, and the motion of the projectile is bounded in the co-moving frame.

To illustrate how these considerations operate in the evaluation of phase stability for an accelerator, we introduce numerical values for the various constants. Parameters for a typical accelerator are $v_0 = 10^3$ cm sec^{-1}, $a = 10^{10}$ cm sec^{-2}, $R = 10$ cm, $l_0 = 5$ cm, and $k = 1$ cm^{-1}. Resonance occurs when Eq. (5.28) is satisfied, or, using Eq. (5.15), when

$$3\, a\, l_0 (R^2 + l_0^2)^{-1} = (2\,\pi\, k\, v_1)^2. \qquad (5.29)$$

This gives $v_1 = 5.5 \times 10^3$ cm sec^{-1} which is well above the initial velocity, v_0. However, using

$$x_1 = (v_0/a)\,(v_1 - v_0) + (1/2\,a)\,(v_1 - v_0)^2, \qquad (5.30)$$

one finds that $x_1 = 1.5 \times 10^{-3}$ cm. Thus, resonance velocity is reached before the projectile has traveled even a small fraction of the 1 cm coil spacing between the first and second lumps. By the time the projectile reaches the 1 cm mark its velocity is $v_1 = (2\,a\,x_1)^{1/2} = 1.4 \times 10^5$ cm sec^{-1}. Checking the resonance condition again shows that $(2\,\pi\, k\, v_1)^2 = 7.9 \times 10^{11}$ sec^{-2}, while $3\, a\, l_0 (R^2 + l_0^2)^{-1}$ is still only 1.2×10^9 sec^{-2}, indicating that the driving frequency already is well above resonance.

Furthermore, at the 1 cm point $(\Delta u)_{max} = k\, v_1^2\, a^{-1} = 2$ so that only now are we entering the region in which it is necessary to consider the resonance problem. [If $(\Delta u)_{max}$ is too small, the projectile does not spend enough time (pass enough lumps) at the velocity v_1 for resonance to build up, even if the resonance condition is satisfied.] Since Eq. (5.27) is increasingly well satisfied at higher velocities, phase stability will be maintained.

One may show that this result was not due merely to a lucky choice of parameters. From Eq. (5.22) the velocity necessary to make $(\Delta u)_{max} > 1$ is $v_1^2 > a/k$. Then the condition Eq. (5.27) written in the form of Eq. (5.29) becomes

$$3\, l_0 (R^2 + l_0^2)^{-1} \ll (2\,\pi)^2\, k. \qquad (5.31)$$

The condition is independent of m and a and is satisfied by a wide range of the variables R, l_0, and k. For a given R the left side of Eq. (5.31) is largest when $l_0 = R$. In that case,

$$3/2 \ll (2\,\pi)^2\, k\, R, \qquad (5.32)$$

which simply says that the coil spacing should be no greater than the coil radius.

VI. Stabilization of the Other Motions of the Projectile

Ideally, the projectile's magnetic moment is colinear with the accelerator axis. We now investigate the stability of the projectile with respect to small rotational or transverse translational displacements.

Let us consider the case in which the projectile is rotated through a small angle Θ with respect to the z-axis, as shown in Fig. 6.1. The torque on the projectile is then given by

$$\boldsymbol{N} = \boldsymbol{M} \times \boldsymbol{B} = (\hat{n} \times \hat{k})\,[M_\varrho\, B_z - M_z\, B_\varrho]. \qquad (6.1)$$

Symmetry requires that $B_\varrho = 0$ on the z-axis, so that

$$\boldsymbol{N} = \hat{l}\, M_\varrho\, B_z. \qquad (6.2)$$

However, $M_\varrho = M \sin\Theta$. Using Eq. (5.3), we write

$$\boldsymbol{N} = \hat{l}\, 2\,\pi\, c^{-1}\, I\, M\, [1 - z(R^2 + z^2)^{-1/2}] \sin\Theta. \qquad (6.3)$$

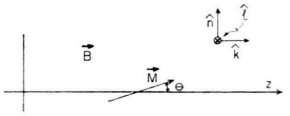

Fig. 6.1.

Since $A_1 > 0$ [see Eq. (5.10)], we have $IM < 0$. The bracket in this last equation is always positive, so that the torque points in a direction opposite to l and therefore leads to instability as it tends to increase Θ. One method of overcoming this instability is to separate the center of mass of the projectile from the center of force by the position vector $\boldsymbol{\lambda}$. The torque on the projectile due to inertial forces is then given by

$$\boldsymbol{N}_I = \boldsymbol{\lambda} \times \boldsymbol{F}, \quad \boldsymbol{F} = -m\,\boldsymbol{a}. \tag{6.4}$$

It is evident that

$$\boldsymbol{\lambda} = -\hat{k}\,\lambda \cos\Theta - \hat{n}\,\lambda \sin\Theta. \tag{6.5}$$

Thus we have

$$\hat{\boldsymbol{N}}_I = \hat{l}\,\lambda\, m\, a \sin\Theta, \tag{6.6}$$

and the total torque is given by

$$\boldsymbol{N}_T = \hat{l} \sin\Theta [\lambda\, m\, a + 2\pi c^{-1} I M (1 - z(R^2 + z^2)^{-1/2})]. \tag{6.7}$$

Proper choices for the parameters results in a positive value for the bracketed term and, hence, produce rotational stability. This is the same principle used in stabilizing a boat against capsizing. Use of Eqs. (5.10) and (5.12) shows that there is rotational stability for $\lambda \geqq R$.

An alternative method for overcoming the rotational instability would be gyroscopic stabilization. In this scheme the spinning projectile would precess about the z-axis and execute small nutations in Θ.

The transverse translational motion is inherently unstable. The proof of this is similar as for Earnshaw's theorem in electrostatics [7]. Methods such as strong focusing or feedback control will be required to stabilize against transverse displacements.

VII. Summary and Remarks

The design of a magnetic travelling wave accelerator for superconducting macroparticles is clearly a sizeable task. Some aspects of the important problems of wave generation and dynamic stability have been considered here. It has been shown that the use of lumped parameters in place of the mathematically simpler continuous parameters will still provide a usable current wave and, within very light restrictions, will not introduce a phase instability. It has also been shown that rotational stability can be achieved by separating the projectile's center of mass and center of force. It was pointed out, however, that the projectile's motion is unstable in the transverse direction.

Finding a workable system for obtaining transverse stability will be an important advance in macroparticle accelerator technology. Some of the remaining problems of great interest are: 1) The effect of the projectile's magnetic field in the driving current wave; one approach is outlined in the Appendix; 2) The effect of mutual inductance in damping the current wave, and 3) The procedure for storing and injecting the projectile.

Appendix

Flux Interaction Current-Wave

To incorporate the effects of a flux interaction on the currents in a TLM, the voltage equation must be modified by using Faraday's law. The voltage difference over a distance Δx is then given by

$$\Delta V = -L_1 I_t \Delta x - (\Phi_1)_t \Delta x, \tag{A 1}$$

where the subscript t denotes a partial derivative with respect to time and $\Phi_1(x,t)$ is the flux per unit length. The sign of the flux term has been chosen so that Φ_1 is positive for an accelerated projectile. The flux may be computed from the relation

$$\Phi_1(x,t) = N(x) \left| \int_S \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{a} \right|, \tag{A 2}$$

where $N(x)$ is the number of turns per unit length of the accelerating coil. The surface integral is taken over any surface through a single coil loop at the position x. The vector $\boldsymbol{B}$ is the magnetic field due to the projectile only.

In the limit $\Delta x \to 0$, the modified TLM equations are

$$-V_x = L_1 I_t + (\Phi_1)_t, \tag{A 3}$$

$$-I_x = C_1 V_t. \tag{A 4}$$

These equations may be combined to give a second order partial differential equation for $I(x,t)$,

$$I_{xx} - I_x \frac{\mathrm{d}}{\mathrm{d}x}(\ln C_1) - I_{tt} L_1 C_1 = C_1 (\Phi_1)_{tt}. \tag{A 5}$$

Taking the Laplace transform of Eq. (A 5) in order to eliminate the time dependence gives

$$i_{xx} + i_x\, g(x) - i\, f(x) = C_1\, \mathcal{L}\{(\Phi_1)_{tt}\}\,, \qquad \text{(A 6)}$$

where $i(x, p) = \mathcal{L}\{I(x, t)\}$, $g(x) = \mathrm{d}(\ln C_1)/\mathrm{d}x$, and $f(x) = p^2 L_1 C_1$. Since Eq. (A 6) is linear in $i(x, p)$, the solution may be written as

$$i(x, p) = i_{\mathrm{w}}(x, p) + i_{\mathrm{f}}(x, p)\,, \qquad \text{(A 7)}$$

with $i_{\mathrm{w}}(x, p)$ satisfying the equations

$$(i_{\mathrm{w}})_{xx} - g(x)\,(i_{\mathrm{w}})_x - f(x)\, i_{\mathrm{w}} = 0\,, \qquad \text{(A 8)}$$

$$i_{\mathrm{w}}(\infty, p) = 0\,, \qquad \text{(A 9)}$$

$$(i_{\mathrm{w}})_x\,(0, p) = C_1(0)\, V_0\,, \qquad \text{(A 10)}$$

and with $i_{\mathrm{f}}(x, p)$ satisfying

$$(i_{\mathrm{f}})_{xx} - g(x)\,(i_{\mathrm{f}})_x - f(x)\, i_{\mathrm{f}} = C_1\, \mathcal{L}\{(\Phi_1)_{tt}\}\,, \qquad \text{(A 11)}$$

$$i_{\mathrm{f}}(\infty, p) = 0\,, \qquad \text{(A 12)}$$

$$(i_{\mathrm{f}})_x\,(0, p) = 0\,. \qquad \text{(A 13)}$$

The subscripts reflect the current source — "w" for wave and "f" for flux interaction.

Using the linearity of the Laplace transform, we obtain for the inverse transform of Eq. (A 7)

$$I(x, t) = I_{\mathrm{w}}(x, t) + I_{\mathrm{f}}(x, t)\,. \qquad \text{(A 14)}$$

The WKB solution to Eq. (A 8), with conditions Eqs. (A 9) and (A 10), has already been given by Eq. (2.19). This current depends only on the initial voltage and the characteristics of the transmission line. All currents from the flux interaction are contained in $I_{\mathrm{f}}(x, t)$.

Equation (A 11) is inhomogeneous but may be solved by the Green's function method. If one puts

$$G_{xx}(x, x') - g(x)\, G_x(x, x') - f(x)\, G(x, x') = C_1\, \delta(x - x')\,, \qquad \text{(A 15)}$$

with

$$G(\infty, x') = 0\,, \qquad \text{(A 16)}$$

$$G_x(0, x') = 0\,, \qquad \text{(A 17)}$$

then

$$i_{\mathrm{f}}(x, \mathrm{p}) = \int_0^\infty \mathrm{d}x'\, G(x, x')\, \mathcal{L}\{(\Phi_1)_{tt}(x', t)\}\,. \qquad \text{(A 18)}$$

The Green's function is determined by finding solutions to the homogeneous form of Eq. (A 15) for $x < x'$ and $x > x'$. These solutions are found by using the result Eq. (2.17) and the boundary conditions Eqs. (A 16) and (A 17). The two arbitrary constants are removed by noting that Eq. (A 15) requires that $G(x, x')$ be continuous at $x = x'$ and have a discontinuity in its first derivative of $C_1(x)$ at $x = x'$. After simplification, the Green's function may be written as

$$G(x, x') = -(1/2\,p) \cdot \{C_1(x)\, C_1(x')/[L_1(x)\, L_1(x')]\}^{1/4} \cdot \{\exp[-p\,|(-)|] + \exp[-p(+)]\}\,, \qquad \text{(A 19)}$$

where

$$(-) = \int_x^{x'} (L_1 C_1)^{1/2}\, \mathrm{d}x\,, \qquad \text{(A 20)}$$

$$(+) = \int_0^x (L_1 C_1)^{1/2}\, \mathrm{d}x + \int_0^{x'} (L_1 C_1)^{1/2}\, \mathrm{d}x\,. \qquad \text{(A 21)}$$

Now $G(x, x')$ may be substituted into Eq. (A 18). Use of the convolution theorem in taking the inverse transform yields

$$I_{\mathrm{f}}(x, t) = -\tfrac{1}{2}\,[C_1(x)/L_1(x)]^{1/4} \times \int_0^\infty \mathrm{d}x' \int_0^t [C_1(x')/L_1(x')]^{1/4} \times \{H[t - t' - |(-)|] + H[t - t' - (+)]\}\,(\Phi_1)_{tt}(x', t')\, \mathrm{d}t'. \qquad \text{(A 22)}$$

Integration of the t' integral by parts gives the alternate expression

$$I_{\mathrm{f}}(x, t) = -\tfrac{1}{2}\,[C_1(x)/L_1(x)]^{1/4} \times \int_0^\infty \mathrm{d}x'\,[C_1(x')/L_1(x')]^{1/4} \Big[\{H[t - t' - |(-)|] + H[t - t' - (+)]\}\,(\Phi_1)_t\,(x', t')\Big|_0^t + \int_0^t \mathrm{d}t'\{\delta[t - t' - |(-)|] + \delta[t - t' - (+)]\}\,(\Phi_1)_{tt}(x', t')\Big]\,. \qquad \text{(A 23)}$$

This equation may serve as the starting point for a numerical analysis of a TLM in which the flux interaction between the projectile and the field coils is included.

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